On the Development of the SIA-Log-Laplace Distribution and its Power Transformation

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***Abstract*—During the past five to six years, some researchers have focused on a particular class of distributions --- those that are now being regarded as Self-Inverse at A (SIA) --- and the self-inversion property has begun to be utilized for developing estimators of distribution parameters that are *more efficient* than the well-known estimators. Efficiency of the sampling distribution implies that there exists a higher probability of *more accurate* modeling, and this fact provides motivation to develop *new* SIA *distributions* or to obtain the SIA-*versions* of distributions that are invariant under the reciprocal transformation. The Log-Laplace distribution *f(x, c)* has only one parameter *c* which can be regarded as the scale parameter of this distribution. This distribution finds applications in economics as well as in the sciences. In this paper, we obtain the SIA-version of the Log-Laplace distribution and derive its fundamental properties including the first four moments, moment ratios and the quantile function. Applying the power transformation to the SIA Log Laplace distribution, we obtain its generalized version for which we adopt the nomenclature "SIA Log-Laplace Power distribution". We are optimistic that these generalizations will assist in widening the scope and applicability of the Log-Laplace distribution.**

Keywords-Self-Inversion at A; SIA Log-Laplace distribution; moments; moment-ratios; quantile function; SIA Log-Laplace Power distribution.

I. INTRODUCTION

Various classes of distributions have attracted the attention of researchers during the past few decades. The class of continuous distributions of non-negative random variables for which the reciprocal of the random variable possesses exactly the same distribution as the original random variable is now known as the class of distributions Self-Inverse at Unity (SIU). (See [1].) This class of distributions possesses the interesting property that the $(1-q)^{\text{th}}$ quantile is the reciprocal of the q^{th} quantile. The median of each such distribution is unity.

A generalization of the concept of self-inversion at unity has also been given in [1]. This wider class of distributions have been called "distributions Self-Inverse at A (SIA)" in [2] where the arbitrary positive number A represents the median of the distribution.

The remarkable property of self-inverse distributions is that the relationship between the upper and lower quantiles creates opportunities to develop estimators of distribution parameters that are *more efficient* than the well-known estimators. Efficiency of the sampling distribution implies that there exists a higher probability of *more accurate* modeling, and this fact provides motivation to develop *new* SIA distributions or to obtain the SIA-*versions* of distributions that are Self-Inverse at Unity**.**

The Log-Laplace distribution is a member of the class of SIU distributions and finds applications in economics as well as in sciences such as particle size of studies. In this paper, we take up the simple form of the Log-Laplace distribution which has been presented in [3] and obtain the SIA-version of this distribution. We obtain the fundamental properties of the newly derived "SIA Log-Laplace distribution" including moments and moment ratios. Applying the power transformation to the "SIA Log Laplace distribution", we obtain the "SIA Log-Laplace Power distribution". It is easy to see that the mean, variance, etc. of the "SIA Log-Laplace distribution" are only *a special case* of the corresponding parameters of the much wider family of "SIA Log-Laplace Power distributions".

It is expected that these generalizations will facilitate the widening of the scope and applicability of the Log-Laplace distribution given in [3].

II. LOG-LAPLACE DISTRIBUTION

The Laplace distribution is a continuous probability distribution which is used in many fields of science as well as in economics. The probability density function $w(y;c)$ of the Laplace distribution function is given by

$$w(y;c) = \frac{c}{2}\exp\left(-c\left|y\right|\right), -\infty < y < \infty, c > 0 \tag{1}$$

As indicated in [3], if y has the Laplace distribution with parameter c then $z = e^{y}$ has the Log-Laplace distribution, the probability density function of which is given by

$$g(z)=\begin{cases}\frac{c}{2}z^{c-1}, & 0<z<1\\ \frac{c}{2}z^{-c-1}, & 1<z<\infty\end{cases} \tag{2}$$

The distribution given by eq. (2) is invariant under the reciprocal transformation i.e. Self-Inverse at Unity (SIU). It possesses the interesting property that the $(1-q)^{\text{th}}$ quantile is the reciprocal of the q^{th} quantile. The median of the distribution is unity.

The cumulative distribution function *F(z)* of the Log-Laplace distribution given by eq. (2) is

$$G(z)=\begin{cases}\frac{1}{2}z^{c}, & 0<z<1\\ 1-\frac{1}{2}z^{-c}, & 1<z<\infty\end{cases} \tag{3}$$

where $c>0$.

The Log-Laplace distribution finds applications in economics as well as in the sciences such as particle size studies.

III. SIA-LOG-LAPLACE DISTRIBUTION:

Applying the transformation $X = AZ, A > 0$ on the density function of the Log-Laplace distribution given by eq. (2), the probability density function of the "SIA-Log Laplace distribution" comes out to be

$$f(x)=\begin{cases}\frac{c}{2A}\left(\frac{x}{A}\right)^{c-1}, & 0<x<A\\ \frac{c}{2A}\left(\frac{x}{A}\right)^{-c-1}, & A<x<\infty\end{cases} \tag{4}$$

As such, the CDF is given by

$$F(x)=\begin{cases}\frac{1}{2}\left(\frac{x}{A}\right)^{c}, & 0<x<A\\ 1-\frac{1}{2}\left(\frac{x}{A}\right)^{-c}, & A<x<\infty\end{cases} \tag{5}$$

The graphs of the PDF and the CDF of the "SIA Log Laplace distribution" for a constant value of c but varying values of A are given in Figures 1 and 2 respectively.

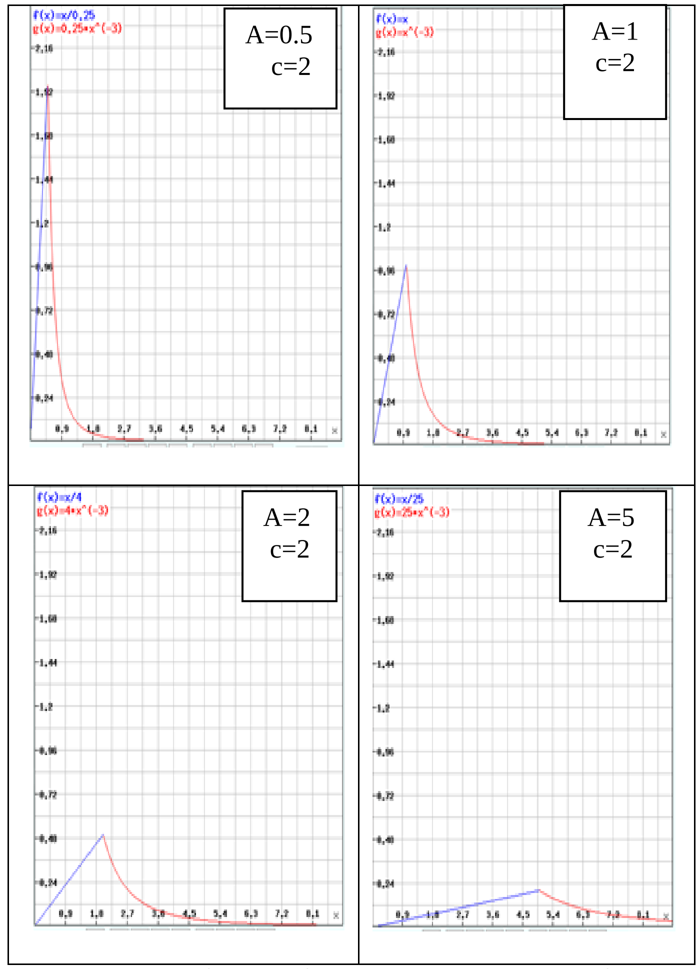

Figure 1. Graph of the PDF of the SIA Log Laplace distribution for a constant value of c but varying values of A.

It is easy to show that the median of the "SIA Log Laplace distribution" is equal to A. Putting A=1, we obtain the Log-Laplace distribution given by eq. (3).

IV. MOMENTS OF THE "SIA LOG-LAPLACE DISTRIBUTION"

In this section, we present the mean, variance, higher moments and moment-ratios of the "SIA-Log Laplace distribution" given by eq. (4). The results are given below.

A. *Mean*

The mean of the distribution is given by

$$E(X)=\frac{Ac^2}{(c^2-1)} \tag{6}$$

From the algebraic expression of the mean, it is obvious that the mean exists only for the case $c>1$.

B. *Variance*

The variance of the distribution is given by

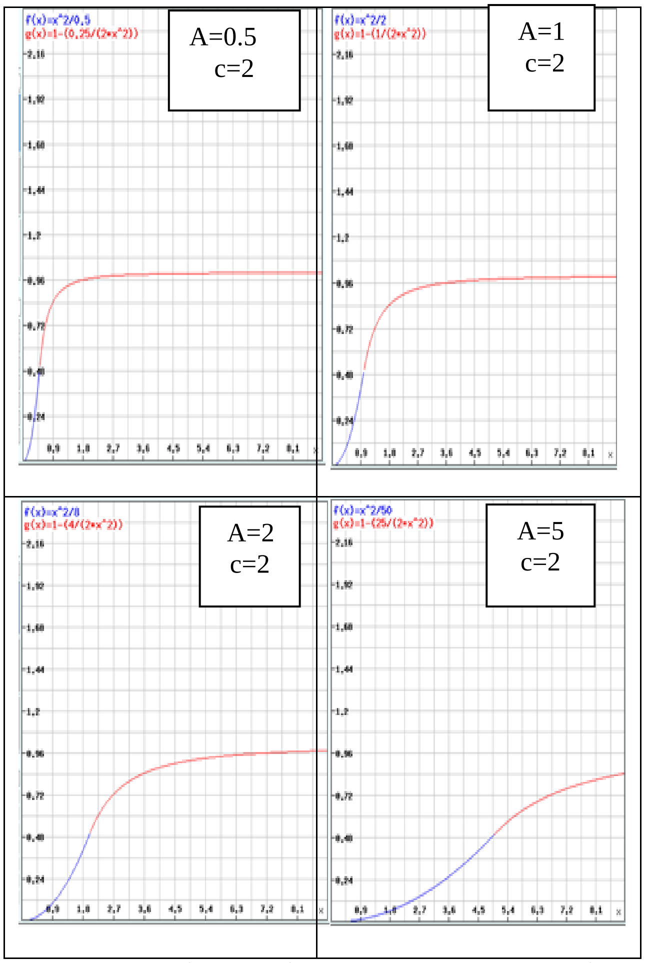

Figure 2. Graph of the CDF of the SIA Log Laplace distribution for a constant value of c but varying values of A.

$$V(X) = \frac{c^2 A^2 [2c^2 + 1]}{(c^2 - 4)(c^2 - 1)^2} \qquad (7)$$

From the algebraic expression of the variance, it is obvious that the variance exists only for the case $c > 2$.

C. *Higher Moments*

Whereas the first two moments provide information regarding the central tendency and dispersion of the distribution, the third and fourth moments are useful in studying the skewness and kurtosis of the distribution.

The 3rd moment of the "SIA-Log-Laplace distribution" is given by

V.

$$\mu_3 = \frac{c^2 A^3 (30c^4 + 14c^2 + 4)}{(c^2 - 9)(c^2 - 4)(c^2 - 1)^3} \qquad (8)$$

for $c > 3$, and the algebraic expression of the 4th moment is

$$\mu_4 = \frac{(488c^8 - 182c^6 + 285c^4 - 413c^2 + 36)}{(c^2 - 16)(c^2 - 9)(c^2 - 4)(c^2 - 1)^4} \qquad (9)$$

where $c > 4$.

D. *Moment ratios*

The moment-ratio β_1 which is a well-known measure of skewness is given by

$$\beta_1 = \frac{(30c^4 + 14c^2 + 4)^2 (c^2 - 4)}{c^2 (c^2 - 9)^2 (2c^2 + 1)^3} \qquad (10)$$

for $c > 3$ whereas the moment ratio β_2 which measures the kurtosis is given by

$$\beta_2 = \frac{(24c^8 + 636c^6 + 285c^4 + 99c^2 + 36)(c^2 - 4)}{c^2 (c^2 - 16)(c^2 - 9)[2c^2 + 1]^2} \qquad (11)$$

for $c > 4$.

Table 1 presents the values of the moment ratios β_1 and β_2 of the "SIA-Log Laplace distribution" given by eq. (4). From the table, it is obvious that, as $c \to \infty$, $\beta_1 \to 0$ and $\beta_2 \to 6$. As such, we conclude that, with an increase in the value of c, the 'SIA-Log Laplace distribution" approaches symmetry but does not become mesokurtic.

TABLE 1 MOMENT RATIOS FOR DIFFERENT VALUES OF THE PARAMETER

c	β_1	β_2
5	9.027706	43.71779
10	1.296876	9.446461
20	0.183946	6.460667
30	0.126891	6.31682
40	0.070907	6.176514
50	0.045243	6.112473
100	0.011265	6.027954
500	0.00045	6.001116
5000	0.0000045	6.000011

V. GEOMETRIC AND HARMONIC MEANS

The Geometric mean of the "SIA-Log Laplace distribution" is identical to its median i.e.

$$G.M. = A \qquad (12).$$

The Harmonic mean of the distribution turns out to be

$$\text{H.M.} = \frac{A\left(c^2 - 1\right)}{c^2} \tag{13}$$

It is interesting to note that the harmonic mean is related to the arithmetic mean by the equation

$$\text{H.M.} = \frac{A^2}{\text{A.M.}} \tag{14}$$

VI. QUANTILE FUNCTION OF THE 'SIA LOG-LAPLACE DISTRIBUTION"

In this section, we derive the quantile function of the "SIA Log-Laplace distribution" given by eq. (4). Since the pdf of the distribution has two components, the first one for values of x ranging from 0 to A and the second one for values of X ranging from A to ∞, therefore the quantile function of this distribution also has two expressions:

(i) For $0 \le p \le 0.5$:

$$x_p = A\left(2p\right)^{\frac{1}{c}} \tag{15}$$

(ii) For $0.5 \le p \le 1$:

$$x_p = A\left(2q\right)^{-1/c} \tag{16}$$

where $q = 1 - p$.

Table 2 contains the values of the first quartile corresponding to different combinations of values of c and A .

TABLE 2 VALUES OF THE FIRST QUARTILE FOR DIFFERENT VALUES OF THE PARAMETERS

$c \setminus A$	**0.2**	**0.5**	**1**	**2**	**5**
0.2	0.0062	0.0156	0.0312	0.0625	0.1562
0.5	0.0500	0.1250	0.2500	0.5000	1.2500
1	0.1000	0.2500	0.5000	1.0000	2.5000
2	0.1414	0.3536	0.7071	1.4142	3.5355
5	0.1741	0.4353	0.8706	1.7411	4.3528

The values of the third quartile corresponding to different combinations of values of c and A are given in Table 3. It is interesting to note that the upper quartiles are related to the lower quartiles by the equation

$$Q_3 = A^2 \,/\, Q_1 \tag{14}$$

TABLE 3 VALUES OF THE THIRD QUARTILE FOR DIFFERENT VALUES OF THE PARAMETERS

$c \setminus A$	**0.2**	**0.5**	**1**	**2**	**5**
0.2	6.4000	16.000	32.000	64.000	160.00
0.5	0.8000	2.0000	4.0000	8.0000	20.000
1	0.4000	1.0000	2.0000	4.0000	10.000
2	0.2828	0.7071	1.4142	2.8284	7.0711
5	0.2297	0.5743	1.1487	2.2974	5.7435

VII. "SIA LOG-LAPLACE POWER DISTRIBUTION"

In the above sections, we have presented the fundamental properties of the "SIA Log-Laplace distribution" given by eq. (4). In this section, we obtain a *generalization* of this distribution by applying the power transformation given by

$$W = X^r,\ r \in \square$$

to the "SIA Log-Laplace" random variable. The transformation yields the following probability density function:

$$f(w) = \begin{cases} \dfrac{c}{2A^c r} w^{\frac{c-r}{r}} & 0 < w < A^r \\ \dfrac{c}{2A^{-c} r} w^{\frac{-c-r}{r}} & A^r < w < \infty \end{cases} \tag{15}$$

It is easy to show that the probability density function given by eq. (15) is Self-Inverse at A (SIA). We call it the "SIA Log-Laplace Power distribution".

VIII. MEAN AND VARIANCE OF THE "SIA LOG-LAPLACE POWER DISTRIBUTION"

A. Mean:

The mean of a non-negative continuous random variable X distributed according to the "SIA Log-Laplace Power distribution" is given by

$$E(X) = \int_0^{A^r} x \frac{c}{2A^c r} x^{\frac{c-r}{r}} dx + \int_{A^r}^{\infty} x \frac{c}{2A^{-c} r} x^{\frac{-c-r}{r}} dx$$

$$= \frac{cA^r}{2(c+r)} + \frac{cA^r}{2(c-r)}$$

or, in other words,

$$E(X) = \frac{c^2 A^r}{(c^2 - r^2)} \tag{16}$$

From the algebraic expression of the mean, it is obvious that the mean exists only for the case $c > r$. By letting $r = 1$ in eq. (20), we obtain the mean of the “SIA Log-Laplace distribution” given by eq. (6).

B. Variance

The variance of a non-negative continuous random variable X distributed according to the “SIA Log-Laplace Power distribution” is derived as follows:

$$E(X^2) = \int_0^{A^r} x^2 \frac{c}{2A^{c}r} x^{\frac{c-r}{r}} dx + \int_{A^r}^{\infty} x^2 \frac{c}{2A^{-c}r} x^{\frac{-c-r}{r}} dx$$

$$= \frac{cA^{2r}}{2(c+2r)} + \frac{cA^{2r}}{2(c-2r)} = \frac{c^2 A^{2r}}{(c^2 - 4r^2)}$$

As such, we obtain

$$V(X) = E\left(X^2\right) - \left[E(X)\right]^2 = \frac{c^2 A^{2r}}{(c^2 - 4r^2)} - \left[\frac{c^2 A^r}{(c^2 - r^2)}\right]^2$$

or

$$V(X) = \frac{c^2 A^{2r} r^2 \left[r^2 + 2c^2\right]}{(c^2 - r^2)^2 (c^2 - 4r^2)} \qquad (17)$$

From the algebraic expression of the variance, it is obvious that the variance exists only for the case $c > 2r$. By letting $r = 1$ in eq. (17), we obtain the variance of the “SIA Log-Laplace distribution” given by eq. (7).

It is easy to show that, similar to the mean and variance, the higher moments of the “SIA Log-Laplace distribution” can be obtained from the corresponding moments of the “SIA Log-Laplace Power distribution” by setting r equal to unity.

VIII. CONCLUDING REMARKS

During the past five to six years, some researchers have focused on a particular class of distributions --- those that are now being regarded as Self-Inverse at A (SIA). The remarkable property of SIA-distributions is that for such distributions it is possible to develop estimators of distribution parameters that are *more efficient* than the well-known estimators. Efficiency of the sampling distribution implies that there exists a higher probability of *more accurate* modeling which is greatly desirable in applications. This fact provides motivation to develop *new* SIA distributions or to obtain the SIA-*versions* of distributions that are Self-Inverse at Unity (SIU)**.**

In this paper, we have taken up the Log-Laplace distribution given in [3] which belongs to the class of SIU distributions. We have obtained the SIA generalization of this distribution and have derived the basic properties including moments and quantiles.

By applying the power transformation to the newly derived distribution, we have generated the “SIA Log-Laplace Power distribution”. For obvious reasons, the class of distributions obtained by applying the power transformation is a *wider* class so that the “SIA Log-Laplace distribution” can be regarded as only *a special case* of the class of “SIA Log-Laplace Power distributions”.

We are optimistic that the generalizations obtained in this paper will widen the scope of the Log-Laplace distribution given in [3] in that SIA-estimators of distribution parameters will be developed which will be *more efficient* than the non-SIA estimators obtained by the method of moments and will lead to *more accurate modeling* of real-life data-sets of non-negative continuous random variables found in engineering and other disciplines.

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